Supplementary Material: Proofs

### S-LMI Covariance

s-LMI with 1 common factor decomposes into following

where by definition of s-LMI is diagonal and and is a column vector of factor loadings constant over time. We also need the covariance of the common factor at both time points. Let be the latent regression coefficient which links the common factor to itself at a previous time point such that , where is independent random term (‘innovation’, ‘error’, ‘disturbance’) with . Assuming standardized common factor such that covariance of the common factor at two subsequent time points is

Now lets look at the covariance matrix from the perspective of strict LMI. A s-LMI model imposes that

where

is a block matrix that sandwiches the covariance matrix of the common factor at both time points.

From the above we see that the strict LMI can only be compatible with any process with stationary covariance, if (assuming is non-zero). (When fitting a s-LMI model this is allowed.) We also see that s-LMI is compatible with non-stationary processes where the covariance is proportional to aligning with previous theoretical analysis where covariance increased over time in a LMI preserving model.

A brief note on notation: We’ll be using simply for the s-LMI residual covariance, since residual covariance is assumed invariant over time .

Using the above auxiliary results we can move to analyse the null hypothesis (hypotheses) of no difference between VAR(1) and s-LMI.

### Proof for spanning (matrix) vector space

* Refinements are necessary, but central ideas should be visible.
* Some proofs might be good to show here and use this as a supplement.

Let . Let be a dimensional matrix of ones, the subindex of which will be omitted in the following. Let be dimensional identity matrix, similarly omitting the subindex in the following. We will first show that the space which these two matrices span contains a set of drift matrices that produce covariance matrices perfectly also explained by S-LMI model.

Let . Let be the positive semidefinite hermitian covariance matrix of random variables observed at different time points . We will omit any special cases that might occur at small due to the factor model being just identifiable at . The cross-covariance is . Note that is effectively summing over the covariances for each variable producing then different values, one for each column of , such that

where . Clearly is a scalar multiple of the covariance.

Define innovations contained in to have and so that the covariance of the innovations is diagonal. We also need that the innovation variances are the same . Since completely now defines covariance of , we have that all variables must have identical covariance for all . This is because defines identical cross-lagged autoregression coefficients for all . Note that also has identical autoregression coefficients and hence the variances are also identical for all . This means that

and hence also

Also see that

where we can see that have coordinates in the space , derivable from the equation above. Using induction this can be proven for which means that all cross covariances for any change in time are also in this space.

Now, since we have that all covariances are the same, then for the S-LMI model with all factor loadings must be the same so that . The residual cross covariances must also be the same, at some , as seen above. (By this point the practical constraint is evident, but will be discussed later, elsewhere. - Sakari) Now we have that the S-LMI imposed cross covariance is

where is the regression coefficient for subsequent and the respective product is for multiple subsequent regressions. is the product of regression coefficients of the latent variable to itsefl at previous timepoints. is a diagonal matix of residual covariances, all of which are the same.

#### General criterion for indistinguishability

Assumptions

We have a set of conditions, and their implications, which must be satisfied, if indistinguishability holds:

Another, more general, criterion for indistinguishability that the subspace of positive semi-definite symmetric matrices must be invariant under . This is because is always positive semi-definite symmetric.

## Generalization to intervention design

We now generalize first the S-LMI, as it is straightforward, and second VAR(1) to an intervention design. We consider the simple case where intervention is targeted at either a) the C-F or b) one symptom in a Network. We begin with a simple intervention where the effect is constant, and move to an intervention where the effect is multiplicative.

The S-LMI within time point covariance is:

Where we have as previously assumed, without loss of generality.

Say we intervene on with an intervention which has an effect independent of , so that regardless of what value has. We use the ‘do’ operator to explicitly note that we change the intervention value. Now the variance of C-F is not changed, and consequently the covariance remains unchanged.

This creates a problem. S-LMI does not recognize effects of such interventions.